THE STANDARD MODEL IN NONCOMMUTATIVE GEOMETRY AND FERMION DOUBLING

J. M. GRACIA-BONDÍA,[1]
B. IOCHUM and T. SCHÜCKER[2]

Abstract

The link between chirality in the fermion sector and (anti-)self-duality in the boson sector is reexamined in the light of Connes' noncommutative geometry approach to the Standard Model. We find it to impose that the noncommutative Yang–Mills action be symmetrized in an analogous way to the Dirac–Yukawa operator itself.

[1]Departamento de Matemáticas, Universidad de Costa Rica, San Pedro 2060, Costa Rica and Departamento de Física Teórica, Universidad de Zaragoza, 50009, Zaragoza, Spain.
[2]Université de Provence and Centre de Physique Théorique, CNRS–Luminy, Case 907, 13288 Marseille, France. E-mail: `iochum@cpt.univ-mrs.fr`, `schucker@cpt.univ-mrs.fr`

1. Introduction

The improved Connes–Lott [1] and Chamseddine–Connes [2, 3, 4] models of noncommutative geometry (NCG) have already yielded action functionals, respectively for elementary particles (tying together the gauge bosons and the Higgs sector) and for elementary particles coupled to gravity. In both kinds of models, the generalization of the Yang–Mills (and gravitational, as the case may be) action is obtained from an operator-theoretic data set: $(\mathcal{A}, \mathcal{H}, D, J, \chi)$, where $\mathcal{A}$ is a suitable noncommutative algebra, replacing the algebra of functions on ordinary spacetime, acting on the linear space $\mathcal{H}$, related to the fermionic content of the theory, D is a generalized Dirac operator, involving both spacetime and internal degrees of freedom, J is a charge conjugation operator (making the triple $(\mathcal{A}, \mathcal{H}, D)$ into a *real* spectral triple or K-cycle) and χ is the chirality.

This reinterpretation of the Standard Model (SM) of fundamental interactions as an unveiling of the (noncommutative) geometric structure of the spacetime has many intriguing aspects. The advantages afforded by the tools of (this variant of) NCG over alternative descriptions have been discussed elsewhere [5]. Nevertheless, Lizzi *et al* [6] —see also [7]— recently pointed out that the Hilbert spaces employed till now in the NCG approach overcount the fermion degrees of freedom. They stated that a "physical" Hilbert space could be obtained by a sort of chiral projection of the Hilbert space pertaining to the product K-cycle of [1, 2]; but then, according to them, when retaining only the contribution of the physical states, the NCG procedure to recover the bosonic part of the action fails.

In [6], as in most NCG papers, calculations are made in the Euclidean framework, whereas conclusions are drawn on the SM Lagrangian, in a different spacetime signature. Therefore, in discussing claims such as the ones made by Lizzi *et al*, it seems a useful start, to clarify the procedure employed in NCG for mimicking the Lorentzian framework: on that we only know of a brief remark in all the pertinent literature [8]. We devote the next section to an account of that procedure, which we feel takes out the force of *some of* the arguments in [6]; also we briefly discuss the variants of a relative sign apparently introduced thereby in the fermion sector of the Lagrangian.

On the other hand, the reasoning by Lizzi *et al* uncovers an interesting link between chirality in the fermion sector and (anti-)self-duality in the boson sector; to wit, projection over states of definite chirality in the source happens to lead to projection onto eigenstates of the signature operator in the gauge fields. It is a deep result of noncommutative geometry [11] that all of the geometrical structure of a Riemannian spin manifold can be recovered from a set of axioms for commutative real spectral triples. A similar theorem for Lorentzian spin manifolds (and thus a *prima facie* Lorentzian formulation of the Connes–Lott and Chamseddine–Connes models of noncommutative geometry) as yet does not exist, and, to our minds, it may demand abandoning the (tensor) *product spectral triples* till now favoured in the literature. At any rate, however, the axioms of algebraic character (in particular, the crucial condition of commutativity between $[D, \mathcal{A}]$ and $J\mathcal{A}J^{\dagger}$) among that set,

can be straightforwardly reformulated in the Lorentzian context; and so can the algebraic constructs leading to the Connes–Lott type of models, provided one is willing to trade Krein spaces for Hilbert spaces and work with pseudoscalar products [5]. Now, the link between chirality and self-duality seems robust enough to be essentially independent of the signature —and there, the arguments of [6] cannot be so easily dismissed.

In effect, we show in section 3 that the troubles pointed out in [6] are a blessing in disguise. The aforementioned link motivates us to plant signposts in the way of the future Lorentzian formulation of the theory, and also helps to decide a relatively subtle question in real spectral triple theory. In the third section of the paper, we show that the bosonic part of the SM Lagrangian can be reconstructed in the Connes–Lott way, even when the chiral projection of [6] is made, provided that the *symmetrized* inner product on noncommutative differential forms (seen as operators on the pertinent linear space) is adopted; i.e., provided one takes into account not only the given representation of $\mathcal{A}$, but also its conjugate representation by J —which Lizzi *et al* omitted to do.

2. Minkowskian versus Euclidean

We start by recalling the noncommutative spectral data for the SM, following [5]. Since the Třešť conference of May 95, it has become standard in the NCG literature to identify the arena for the SM as the algebra

$$\mathcal{A}_t := C^\infty(M, \mathbb{R}) \otimes \mathcal{A}_f,$$

where M denotes the ordinary spacetime and $\mathcal{A}_f$ the finite dimensional real Eigenschaften algebra:

$$\mathcal{A}_f := \mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C}),$$

involving in particular the algebra $\mathbb{H}$ of quaternions.

In the product spectral triple formulation, the Hilbert space $\mathcal{H}_t$ is the tensor product of a generic *bispinor* space $L^2(S_M)$ by a 90-dimensional complex space $\mathcal{H}_f$ with a basis labelled by all chiral particles and antiparticles in the SM; hence the apparent redundancy of degrees of freedom. An element of $\mathcal{H}$ is written as a multispinor field $\begin{pmatrix} \Psi_L \\ \Psi_R \\ \Psi_L^c \\ \Psi_R^c \end{pmatrix}$. We have indicated that $\mathcal{A}$ acts on $\mathcal{H}$. The (real) representation π of $\mathcal{A}$ in the lepton sector is given by

$$\pi(q, \lambda, m) \begin{pmatrix} \Psi_L \\ \Psi_R \\ \Psi_L^c \\ \Psi_R^c \end{pmatrix} = \begin{pmatrix} q\Psi_L \\ \lambda\Psi_R \\ \bar{\lambda}\Psi_L^c \\ \bar{\lambda}\Psi_R^c \end{pmatrix}.$$

In the quark sector:

$$\pi(q, \lambda, m) \begin{pmatrix} \Psi_L \\ \Psi_R \\ \Psi_L^c \\ \Psi_R^c \end{pmatrix} = \begin{pmatrix} q\Psi_L \\ \pi^q(\lambda)\Psi_R \\ m\Psi_L^c \\ m\Psi_R^c \end{pmatrix},$$

where m acts on the internal color space of each quark field and π^q is the real representation of $C^\infty(M, \mathbb{C})$ acting as $\bar\lambda$ on the u-labelled quark fields and as λ on the d-labelled quark fields. Moreover, if we write $D_t = \partial\!\!\!/ \otimes 1 + \gamma_5 \otimes D_f$, $J_t = C \otimes J_f$, where C denotes charge conjugation, and $\chi_t = \gamma_5 \otimes \chi_f$ where χ_f is the natural grading on $\mathcal{H}_f$, then $(\mathcal{A}_t, \mathcal{H}_t, D_t, J_t, \chi_t)$ is a real spectral triple.

We wish to remark [9] that the quaternions are already present in the usual formalism of the Standard Model. This is tantamount to the $SU(2)$ gauge invariance of the mass terms —see, for instance [10]. The fact is not always recognized or exploited.

The "covariant" Dirac operator, restricted to the particle space, is schematically written:

$$D_p = \begin{pmatrix} \gamma^\mu(i\partial_\mu + L_\mu) & \gamma_5 M \\ \gamma_5 M^\dagger & \gamma^\mu(i\partial_\mu + R_\mu) \end{pmatrix},$$

where, for instance in the quark sector,

$$M = \begin{pmatrix} \bar\phi^0 & \phi^+ \\ -\phi^- & \phi^0 \end{pmatrix} \begin{pmatrix} m_u & \\ & m_d \end{pmatrix},$$

with the 3×3 (where 3 is the number of fermionic generations and we are forgetting here about color) matrices m_u, m_d encoding all the dimensionless Yukawa couplings and Kobayashi–Maskawa mixing parameters, and the mass matrix M_0 could be exhibited by shifting ϕ^0, $\bar\phi^0$ by a constant with the dimensions of the scalar field (i.e., mass), corresponding to its vacuum expectation value. The L_μ and R_μ collectively denote the gauge fields coupled to each chiral sector, which *together with the Higgs field*, in NCG are determined by the interplay between the Dirac operator and the world algebra action: the full Dirac–Yukawa operator on the space of particles and antiparticles is $D = D_p + JD_pJ^\dagger$ and the coupling of fermions to all boson fields is described by $\mathbb{A} + J\mathbb{A}J^\dagger$, where $\mathbb{A}$ denotes the noncommutative gauge potential (a Hermitian, noncommutative 1-form) associated to the triple $(\mathcal{A}, \mathcal{H}, D)$. In order that our chosen data set $(\mathcal{A}, \mathcal{H}, D, J)$ be accepted as a certified real spectral triple, one needs to check that both $\mathcal{A}$ and $[D, \mathcal{A}]$ commute with the "conjugate action" $J\mathcal{A}J^\dagger$ and reciprocally [11].

Note that the choice of D_t is not unique; for instance, $D_t = U(\partial\!\!\!/ \otimes \chi_f + 1 \otimes D_f)U^\dagger$, where U is the unitary $U = P_+ \otimes 1 + P_- \otimes \chi_f$ with $P_\pm = \frac{1}{2}(1 \pm \gamma_5)$ is another possible choice. This shows that the relative sign between the mass terms introduced by the presence of γ_5 in D_t can be replaced by another one between kinetic terms when the internal chirality χ_f is prefered and has nothing to do with a choice of signature.

Let us briefly review the old story of chiral fermion doubling in the Euclidean. In the Minkowskian, for one species of fermions, nature has elected the Dirac Lagrangian,

$$\overline{\Psi_L}\gamma^\mu\, i\partial_\mu\Psi_L + \overline{\Psi_R}\gamma^\mu\, i\partial_\mu\Psi_R,$$

where Ψ_L and Ψ_R are two component chiral spinors,

$$\Psi_L := \frac{1-\gamma_5}{2}\,\Psi, \quad \Psi_R := \frac{1+\gamma_5}{2}\,\Psi,$$

and the pseudoscalar product $\overline{\Psi}\Psi$ is defined by $\overline{\Psi} := \Psi^\dagger\gamma^0$. This pseudoscalar product is motivated physically by probability conservation, mathematically by invariance under the Clifford algebra. For concreteness we specify our Dirac matrices:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3,$$

with the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Dirac matrices satisfy the anticommutation relations,

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}1_4, \quad \eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1).$$

Chirality is then the unitary operator of unit square

$$\gamma_5 = \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \epsilon_{0123} = 1.$$

It anticommutes with all four γ^μ. Charge conjugation is the antiunitary operator of unit square

$$J\Psi = \Psi^c = i\gamma^2\Psi^*.$$

It anticommutes with all four γ^μ.

The invariant scalar product in the Euclidean $\Psi^\dagger\Psi$ has no γ^0 and the corresponding Euclidean Lagrangian,

$$(\Psi_L)^\dagger\gamma^\mu\, i\partial_\mu\Psi_L + (\Psi_R)^\dagger\gamma^\mu\, i\partial_\mu\Psi_R,$$

vanishes identically. The way out is to assume that Ψ_L and Ψ_R are two independent 4-spinors. This doubling of degrees of freedom is usually taken as an artifact of the Euclidean formulation and at the end of day one "Wick rotates" back to the Minkowskian and then imposes the chirality conditions,

$$\frac{1-\gamma_5}{2}\,\Psi_L = \Psi_L, \quad \frac{1+\gamma_5}{2}\,\Psi_R = \Psi_R.$$

In [7], Lizzi *et al* propose to grant physical significance to the additional "mirror" fermions.

3. The physical inner product

The noncommutative integral allows one to define scalar or pseudoscalar products on the algebra of noncommutative differential forms. The prototype would be

$$\fint S^\dagger T := \mathrm{Tr}_{\mathrm{Dix}}\left(S^\dagger T D^{-4}\right),$$

where $\mathrm{Tr}_{\mathrm{Dix}}$ denotes the Dixmier trace. It is known that, under suitably regularity conditions, $\fint$ defines a trace on that algebra [12, 13]; it resolves in our cases into a combination of finite traces and ordinary integrals [5]; the final integrand is to be interpreted as the Lagrangian.

Let $\mathbb{F} = \mathbb{F}^\dagger$ denote the noncommutative gauge field associated to $\mathbb{A}$ (for our algebra $\mathcal{A}_t$) and let P_+ be the projection over the "physical subspace", in the language of [6]. That is, P_+ is equivalent to $\frac{1}{2}(1 + \gamma_5)$ on the subspaces of H_f labelled "right" and to $\frac{1}{2}(1 - \gamma_5)$ on the subspaces of H_f labelled "left". Then the main contention of that paper is that $\fint P_+ \mathbb{F}^2$ yields in that integrand only the anti-self-dual component of the kinetic term of the $SU(2)$ field:

$$C_F\ \mathrm{tr}(F^{\mu\nu} F_{\mu\nu} - F^{\mu\nu} {*}F_{\mu\nu}),$$

where, as usual, ${*}F_{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\sigma\rho} F_{\sigma\rho}$. Analogously, for the $U(1)$ field they get the self-dual component:

$$C_B(B^{\mu\nu} B_{\mu\nu} + B^{\mu\nu} {*}B_{\mu\nu}).$$

For the color field, the usual kinetic term is obtained. Note that the corresponding classical Yang–Mills equations of motion are not modified by the addition of the topological term; but the result is presumably not acceptable on quantum grounds.

With some care, it is feasible to recast the Connes-Lott procedure in the Lorentzian mould. We give now a parallel account of the procedure in [6].

Let $F = \frac{1}{2}F_{\mu\nu}\gamma^\mu\gamma^\nu$ be a Hermitian 2-form. The starting point is the scalar product in component form:

$$\fint F^2 = \frac{1}{128\pi^2}\int_M F_{\mu\nu} F_{\rho\sigma}\ \mathrm{tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = \frac{1}{16\pi^2}\int_M F^{\mu\nu} F_{\mu\nu} = \frac{1}{8\pi^2}\int_M F {*}F.$$

Motivated from the chiral projection $P_\pm$ in the fermionic sector, Lizzi *et al* also project in the bosonic sector. With

$$\fint \gamma_5 F^2 = \frac{1}{128\pi^2}\int_M F_{\mu\nu} F_{\rho\sigma}\ \mathrm{tr}[\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = \frac{i}{32\pi^2}\int_M F_{\mu\nu} F_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} = \frac{i}{8\pi^2}\int_M FF,$$

we get the Yang–Mills action plus the well known topological term

$$\fint (P_\pm F)^2 = \fint P_\pm F^2 = \frac{1}{32\pi^2}\int_M (F^{\mu\nu} F_{\mu\nu} \pm iF^{\mu\nu} {*}F_{\mu\nu}) = \frac{1}{16\pi^2}\int_M (F{*}F \pm iFF).$$

This result is neither unsatisfactory nor wholly unexpected. The situation is reminiscent of the fact that the chiral projection applied to the gravitational action in Schwinger's formulation [14] produces the gravitational action in Ashtekar's formulation [15]. This is common folklore, but we sketch the proof below, because it is not often recognized or exploited by noncommutative geometers. Just write the (Minkowskian) Dirac operator in components,

$$D = \gamma^a e_a^\mu (\partial_\mu + \tfrac{1}{4}\omega_\mu), \quad \omega_\mu := \tfrac{1}{2}\omega_{ab\mu}[\gamma^a, \gamma^b],$$

where $\omega_{ab\mu}$ are the components of the Levi-Civita connection with respect to the orthonormal frame $e_a^\mu \partial_\mu$. After a $1+3$ split, $a = (0, j) = (0, 1, 2, 3)$, we have

$$\omega_\mu = -i\epsilon_{0ij}{}^k \begin{pmatrix} {}^{(+)}\chi_{ij\mu}\sigma_k & 0 \\ 0 & {}^{(-)}\chi_{ij\mu}\sigma_k \end{pmatrix},$$

where

$$ {}^{(\pm)}\chi_{ij\mu} = \omega_{ij\mu} \mp i\epsilon_{0ij}{}^\ell \omega_{0\ell\mu} $$

are Ashtekar's variables.

The foregoing is related to the fact that the Lorentz group and the special orthogonal group in three complex dimensions are locally isomorphic, that again seems also widely ignored. One would expect, as suggested at the end of reference [16] in the context of effective actions, that the Chamseddine–Connes action computed from a truly chiral theory would yield an (anti-)self-dual formulation of gravity.

Now, it stands to reason that, in such a context, the appropriate J is the charge conjugation operator for the fermion multiplet of the SM, with suitable phase factors [17]. The crucial observation, from the viewpoint of this paper, is that in the Minkowskian J interchanges particles of *opposite* chirality, that is, $(\Psi_L)^c = (\Psi^c)_R$ is a right-handed spinor and so on.

We assert that, in the same way as the covariant Dirac operator incorporates the symmetrized noncommutative gauge potential $\mathbb{A} + J\mathbb{A}J^\dagger$, the correct inner product formula defining the generalized Yang–Mills functional incorporates the symmetrized noncommutative gauge field:

$$I_{YM}(\mathbb{F}) := \fint (\mathbb{F} + J\mathbb{F}J^\dagger)^2.$$

If $\mathcal{A}$ is commutative, then $J\mathbb{A}J^\dagger = -\mathbb{A}^\dagger$. Thus, for a Hermitian $\mathbb{A}$, we have $\mathbb{A} + J\mathbb{A}J^\dagger = 0$ in that case. However, for the SM, this scalar product is nondegenerate. The symmetrization process is justified by the spectral principle of [2]: $D_t + \mathbb{A}$ is forbidden since for any unitary $u \in \mathcal{A}_t$, the inner automorphism $\alpha_u : a \in \mathcal{A}_t \mapsto uau^\dagger \in \mathcal{A}_t$ induces a unitary operator $U = \pi_t(u)J\pi_t(u)J^\dagger$ satisfying $U\pi_t(a)U^\dagger = \pi_t(\alpha_u(a))$ and $UD_tU^\dagger = D_t + \mathbb{A} + J\mathbb{A}J^\dagger$ with $\mathbb{A} = \pi_t(u)[D_t, \pi_t(u)^\dagger]$. So D_t and $D_t + \mathbb{A} + J\mathbb{A}J^\dagger$ have the same spectrum.

Also, were one to proceed otherwise, one would be treating the contributions of the direct and the conjugate representation (and then of particles and antiparticles) on a different footing, in spite of the fact that their rôles are interchangeable.

We show that, due to the Minkowskian relation $\gamma_5 J = -J\gamma_5$, the pertinent contributions of the $J\mathbb{F}^2 J^\dagger$ term are of the form $C_F \ \mathrm{tr}(F^{\mu\nu}F_{\mu\nu} + F^{\mu\nu}{*}F_{\mu\nu}) + C_B(B^{\mu\nu}B_{\mu\nu} - B^{\mu\nu}{*}B_{\mu\nu})$, matching the previous ones in such a way that the usual SM action is recovered. Indeed,

$$\fint (P_+F + JP_+FJ^\dagger)^2 = \fint (P_+F)^2 + \fint (P_-F)^2 + 0 = \fint F^2.$$

To summarize: Connes' spectral principle makes the symmetrized scalar product mandatory. Lizzi *et al* uncovered an independent argument in favour of this symmetrized product:

$$\langle S^\dagger \mid T \rangle := \Re e \fint (S + JSJ^\dagger)^\dagger (T + JTJ^\dagger)$$

on the *real* differential algebra of the real spectral triples.

Besides the SM, in [6] the noncommutative bosonic action is computed for another model, in which there are two different $SU(2)$ gauge fields, each one coupling to each chiral sector. They as well obtain, in their Euclidean framework, both in the Connes–Lott and in the Chamseddine–Connes temperaments, that the projection over states of definite chirality on the fermionic source leads in NCG to a similar projection in the gauge fields. The results of Lizzi *et al* for that toy model of course stand, but let us observe that it is not endowed with a charge conjugation operator —and in that sense perhaps does not describe a true noncommutative geometry.

Our argument hinges on the fact that antiparticles have the opposite chirality to their corresponding particles. It stands for the present in a sort of no man's land, as the just-mentioned property is of course murky in the Euclidean formalism, the only one in which rigorous NCG calculations to date have been performed: to give a fully Lorentzian version of the NCG action principle is a formidable task, that we leave for another day.

The NCG models of the general type considered in this paper give constrained versions of the SM, i.e., with some relations among the parameters of the boson sector, allowing, namely, to make educated guesses on the Higgs particle mass. Those coefficients vary when computed with the symmetrized inner product. The computation is not entirely straightforward, chiefly because of the well known nonlinearity that arises (in the Connes–Lott procedure) when combining the contribution of the quark and the lepton sectors. That issue has no bearing on the main point of this article and has been reported separately [18].

Acknowledgments

References

[1] A. Connes, J. Math. Phys. **36**, 6194 (1995).

[2] A. H. Chamseddine and A. Connes, Phys. Rev. Letters **77**, 4868 (1996).

[3] L. Carminati, B. Iochum, D. Kastler and T. Schücker, On Connes' new principle of general relativity. Can spinors hear the forces of spacetime?, Proc. Operator Algebras and Quantum Field Theory, AMS Publications, to appear.

[4] H. Figueroa, F. Lizzi, J. M. Gracia-Bondía and J. C. Várilly, A nonperturbative form of the spectral action principle in Noncommutative Geometry, hep-th/9701179, J. Geom. Phys., to appear.

[5] C. P. Martín, J. M. Gracia-Bondía and J. C. Várilly, The Standard Model as a noncommutative geometry: the low energy regime, hep-th/9605001, Phys. Rep., in press.

[6] F. Lizzi, G. Mangano, G. Miele and G. Sparano, Fermion Hilbert space and fermion doubling in the Noncommutative Geometry approach to gauge theories, hep-th/9610035, Phys. Rev. D **55**, 6357 (1997).

[7] F. Lizzi, G. Mangano, G. Miele and G. Sparano, Mirror fermions in noncommutative geometry, hep-th/9704184.

[8] A. Connes and J. Lott, The metric aspect of noncommutative geometry, in Proceedings of the Cargèse Summer Conference, J. Fröhlich et al, eds. (Plenum Press, London, 1992).

[9] G. Cammarata and R. Coquereaux, in *Low-dimensional models in statistical physics and quantum field theory*, H. Grosse and L. Pittner, eds. (Lecture Notes in Physics 469, Springer, Berlin, 1996).

[10] K. Huang, *Quarks, Leptons and Gauge Fields*, 2nd edition (World Scientific, Singapore, 1992).

[11] A. Connes, Brisure de symétrie spontanée et géométrie du point de vue spectral, Séminaire Bourbaki, 48ème année, No. 816 (1995–1996).

[12] M. Walze, Ph. D. Thesis, Universität Mainz, 1996.

[13] F. Cipriani, D. Guido and S. Scarlatti, J. Oper. Theory **35**, 179 (1996).

[14] J. Schwinger, Phys. Rev. **130**, 1253 (1963).

[15] A. Ashtekar, Phys. Rev. Lett. **57**, 2244 (1986).

[16] L. N. Chang and C. Soo, Phys. Rev. D **55**, 2410 (1997).

[17] F. Scheck, *Electroweak and strong interactions* (Springer, Berlin, 1996).

[18] L. Carminati, B. Iochum and T. Schücker, Noncommutative Yang–Mills and noncommutative relativity: a bridge over troubled water, hep-th/9706105.